The Identical Relations between the Elements of any Oblique Triple System of Surfaces.

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The Lamé identical relations between the elements of an orthogonal triple system of surfaces, are particular cases of six identical relations holding between the elements of a general oblique triple system of surfaces.

Consider a triple system of surfaces, with the parameters ρ_1, ρ_2, ρ_3; and let $M(\rho_1, \rho_2, \rho_3)$ be any point in space, and M_1, M_2, M_3, M_{123} be $(\rho_1+d\rho_1, \rho_2, \rho_3)$, $(\rho_1, \rho_2+d\rho_2, \rho_3)$, $(\rho_1, \rho_2, \rho_3+d\rho_3)$, $(\rho_1+d\rho_1, \rho_2+d\rho_2, \rho_3+d\rho_3)$, respectively, etc.; let x_{ij} represent the derivative with respect to ρ_i and ρ_j, and the summation Σx represent the sum of like terms in x, y, z. And let

$$H_i^2 = \Sigma x_i^2; \qquad k_{ij} = \Sigma x_i x_j. \tag{A}$$

Let $\partial H_i/\partial\rho_j$ be represented by H_{ij}, $\partial k_{ij}/\partial\rho_l$ by k_{ijl}, etc.; so that the subscripts after the first for the capital letters H, and after the second for the lower case letters k, represent direct differentiation. Represent $k_{123}+k_{231}+k_{312}$ by $2s$; and $\partial s/\partial\rho_i$ by s_i. With this notation the differentiation of the equations (A) gives, after certain combinations in some cases, the following eighteen equations (B)

$$\left.\begin{aligned}\Sigma x_1 x_{11} &= H_1 H_{11}\\ \Sigma x_2 x_{11} &= k_{121} - H_1 H_{12}\\ \Sigma x_3 x_{11} &= k_{131} - H_1 H_{13}\end{aligned}\right\}(\mathrm{B}_{11}) \quad \left.\begin{aligned}\Sigma x_1 x_{22} &= k_{122} - H_2 H_{21}\\ \Sigma x_2 x_{22} &= H_2 H_{22}\\ \Sigma x_3 x_{22} &= k_{232} - H_2 H_{23}\end{aligned}\right\}(\mathrm{B}_{22}) \quad \left.\begin{aligned}\Sigma x_1 x_{33} &= k_{133} - H_3 H_{31}\\ \Sigma x_2 x_{33} &= k_{233} - H_3 H_{32}\\ \Sigma x_3 x_{33} &= H_3 H_{33}\end{aligned}\right\}(\mathrm{B}_{33})$$

$$\left.\begin{aligned}\Sigma x_1 x_{23} &= s - k_{231}\\ \Sigma x_2 x_{23} &= H_2 H_{23}\\ \Sigma x_3 x_{23} &= H_3 H_{32}\end{aligned}\right\}(\mathrm{B}_{23}) \quad \left.\begin{aligned}\Sigma x_1 x_{13} &= H_1 H_{13}\\ \Sigma x_2 x_{13} &= s - k_{132}\\ \Sigma x_3 x_{13} &= H_3 H_{31}\end{aligned}\right\}(\mathrm{B}_{13}) \quad \left.\begin{aligned}\Sigma x_1 x_{12} &= H_1 H_{12}\\ \Sigma x_2 x_{12} &= H_2 H_{21}\\ \Sigma x_3 x_{12} &= s - k_{123}\end{aligned}\right\}(\mathrm{B}_{12})$$

The first derivatives of the equations (B), added and subtracted in such a way that the terms containing a third derivative of x, y, z are eliminated, give exactly six relations, namely:

$$H_2 H_{233} + H_3 H_{322} + H_{23}^2 + H_{32}^2 - k_{2323} = \Sigma x_{23}^2 - \Sigma x_{22} x_{33} \tag{G$_1$}$$
$$H_3 H_{311} + H_1 H_{133} + H_{31}^2 + H_{13}^2 - k_{3131} = \Sigma x_{31}^2 - \Sigma x_{33} x_{11} \tag{G$_2$}$$
$$H_1 H_{122} + H_2 H_{211} + H_{12}^2 + H_{21}^2 - k_{1212} = \Sigma x_{12}^2 - \Sigma x_{11} x_{22} \tag{G$_3$}$$

$$H_1 H_{123} + H_{12} H_{13} + k_{2311} - s_1 = \Sigma x_{12} x_{13} - \Sigma x_{11} x_{23} \tag{C$_1$}$$
$$H_2 H_{213} + H_{21} H_{23} + k_{3122} - s_2 = \Sigma x_{23} x_{21} - \Sigma x_{22} x_{31} \tag{C$_2$}$$
$$H_3 H_{312} + H_{31} H_{32} + k_{1233} - s_3 = \Sigma x_{31} x_{32} - \Sigma x_{33} x_{12} \tag{C$_3$}$$

The terms on the right of the equations (G) and (C) can be obtained in terms of the first derivatives of the expressions H and k as follows: Let (x_1, y_1, z_1), (x_{23}, y_{23}, z_{23}), etc. be considered as proportional to the direction cosines of lines named l_1, l_{23}, etc., and represent Σx_{23}^2 by L_{23}^2, etc.; then the cosines of the angles between l_1, l_2, l_3 and l_{23}, l_{31}, l_{12}; l_{11}, l_{22}, l_{33} are given by the equations (B) each divided by the proper H and L; namely, for example, cos (l_2, l_{23}) is H_2H_{23}/H_2L_{23}, etc. The identical relation between two systems of four directions in space (cf. Pascal-Timerding, *Repertorium der höheren Geom.*, 2d ed., 1910, p. 74), applied to the directions l_1, l_2, l_3, l_{22} as the elements of the first system and l_1, l_2, l_3, l_{33} as the elements of the second system, gives

$$0=\begin{vmatrix} 1 & \dfrac{k_{12}}{H_1H_2} & \dfrac{k_{13}}{H_1H_3} & \dfrac{k_{122}-H_2H_{21}}{H_1L_{22}} \\ \dfrac{k_{12}}{H_1H_2} & 1 & \dfrac{k_{23}}{H_2H_3} & \dfrac{H_2H_{22}}{H_2L_{22}} \\ \dfrac{k_{13}}{H_1H_3} & \dfrac{k_{23}}{H_2H_3} & 1 & \dfrac{k_{232}-H_2H_{23}}{H_3L_{22}} \\ \dfrac{k_{133}-H_3H_{31}}{H_1L_{33}} & \dfrac{k_{233}-H_3H_{32}}{H_2L_{33}} & \dfrac{H_3H_{33}}{H_3L_{33}} & \dfrac{\Sigma x_{22}\cdot x_{33}}{L_{22}L_{33}} \end{vmatrix},$$

or

$$0=\begin{vmatrix} H_1^2 & k_{12} & k_{13} & k_{122}-H_2H_{21} \\ k_{12} & H_2^2 & k_{23} & H_2H_{22} \\ k_{13} & k_{23} & H_3^2 & k_{232}-H_2H_{23} \\ k_{133}-H_3H_{31} & k_{233}-H_3H_{32} & H_3H_{33} & \Sigma x_{22}x_{33} \end{vmatrix}.$$

Use V as a symbol for

$$\begin{vmatrix} H_1^2 & k_{12} & k_{13} \\ k_{12} & H_2^2 & k_{23} \\ k_{13} & k_{23} & H_3^2 \end{vmatrix};$$

then the identical relation above is

$$\Sigma x_{22}x_{33}=\frac{-1}{V}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & k_{122}-H_2H_{21} \\ k_{12} & H_2^2 & k_{23} & H_2H_{22} \\ k_{13} & k_{23} & H_3^2 & k_{232}-H_2H_{23} \\ k_{133}-H_3H_{31} & k_{233}-H_3H_{32} & H_3H_{33} & 0 \end{vmatrix}. \qquad (\mathrm{D}_1)$$

In precisely the same way, it can be proved that

$$\Sigma x_{23}^2=\frac{-1}{V}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & s-k_{231} \\ k_{12} & H_2^2 & k_{23} & H_2H_{23} \\ k_{13} & k_{23} & H_3^2 & H_3H_{32} \\ s-k_{231} & H_2H_{23} & H_3H_{32} & 0 \end{vmatrix}, \qquad (\mathrm{D}_1')$$

$$\Sigma x_{12} x_{13} = \frac{-1}{V} \begin{vmatrix} H_1^2 & k_{12} & k_{13} & H_1 H_{12} \\ k_{12} & H_2^2 & k_{23} & H_2 H_{21} \\ k_{13} & k_{23} & H_3^2 & s - k_{123} \\ H_1 H_{13} & s - k_{132} & H_3 H_{31} & 0 \end{vmatrix},$$

$$\Sigma x_{11} x_{23} = \frac{-1}{V} \begin{vmatrix} H_1^2 & k_{12} & k_{13} & H_1 H_{11} \\ k_{12} & H_2^2 & k_{23} & k_{121} - H_1 H_{12} \\ k_{13} & k_{23} & H_3^2 & k_{131} - H_1 H_{13} \\ s - k_{231} & H_2 H_{23} & H_3 H_{32} & 0 \end{vmatrix}.$$

The eight other terms on the right of the equations (G) and (C) can be obtained from these by cyclic permutation of the subscripts.

The equations (D_1), (D_1'), for example, can be easily changed to the form

$$\Sigma x_{22} x_{33} = \frac{1}{V \begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}} \begin{vmatrix} k_{122} - H_2 H_{21} & k_{12} & k_{13} \\ H_2 H_{22} & H_2^2 & k_{23} \\ k_{232} - H_2 H_{23} & k_{23} & H_3^2 \end{vmatrix} \cdot \begin{vmatrix} k_{133} - H_3 H_{31} & k_{12} & k_{13} \\ k_{233} - H_3 H_{32} & H_2^2 & k_{23} \\ H_3 H_{33} & k_{23} & H_3^2 \end{vmatrix}$$

$$- \frac{1}{\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}} \begin{vmatrix} 0 & k_{233} - H_3 H_{32} & H_3 H_{33} \\ H_2 H_{22} & H_2^2 & k_{23} \\ k_{232} - H_2 H_{23} & k_{23} & H_3^2 \end{vmatrix}, \quad (E_1)$$

$$\Sigma x_{23}^2 = \frac{1}{V \begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}} \begin{vmatrix} s - k_{231} & k_{12} & k_{13} \\ H_2 H_{23} & H_2^2 & k_{23} \\ H_3 H_{32} & k_{23} & H_3^2 \end{vmatrix}^2 - \frac{1}{\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}} \begin{vmatrix} 0 & H_2 H_{23} & H_3 H_{32} \\ H_2 H_{23} & H_2^2 & k_{23} \\ H_3 H_{32} & k_{23} & H_3^2 \end{vmatrix}. \quad (E_1')$$

Since by bordering, from (D_1), can be obtained

$$\Sigma x_{22} x_{33} = \frac{-1}{V \begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}} \begin{vmatrix} H_1^2 & k_{12} & k_{13} & k_{122} - H_2 H_{21} & 0 & 0 \\ k_{12} & H_2^2 & k_{23} & H_2 H_{22} & 0 & 0 \\ k_{13} & k_{23} & H_3^2 & k_{232} - H_2 H_{23} & 0 & 0 \\ k_{133} - H_3 H_{31} & k_{233} - H_3 H_{32} & H_3 H_{33} & 0 & k_{233} - H_3 H_{32} & H_3 H_{33} \\ k_{12} & H_2^2 & k_{23} & H_2 H_{22} & H_2^2 & k_{23} \\ k_{13} & k_{23} & H_3^2 & k_{232} - H_2 H_{23} & k_{23} & H_3^2 \end{vmatrix},$$

and this expanded gives (E_1). And so also for all the other terms on the right of the equations (G) and (C).

The equations (G) and (C) are the identical relations between the second derivatives of the elements of a general oblique system of surfaces. On the right of these expressions the first derivatives only appear, in the form given

by the equations (D) or (E). And when $k_{12}=k_{13}=k_{23}=0$, these identical relations reduce to the relations given by Lamé, *Coord. Curv.*, Paris, 1859, (8) p. 76, (9) p. 78. For each of the three surfaces of the triple system, one of the equations (G) is the Gauss relation given in *Disq. gen. c. superficies curvas*, Got., Oct. 8, 1827, p. 119, Art. 11. This is seen at once; since, for example, taking a surface $\rho_1=$const., by a method analogous to that used above in deriving the equation (D_1) the Gauss D, D', D'' are

$$D=\frac{1}{V^{\frac{1}{2}}}\begin{vmatrix} k_{122}-H_2H_{21} & k_{12} & k_{13} \\ H_2H_{22} & H_2^2 & k_{23} \\ k_{232}-H_2H_{23} & k_{23} & H_3^2 \end{vmatrix},$$

$$D'=\frac{1}{V^{\frac{1}{2}}}\begin{vmatrix} s-k_{231} & k_{12} & k_{13} \\ H_2H_{23} & H_2^2 & k_{23} \\ H_3H_{32} & k_{23} & H_3^2 \end{vmatrix},$$

$$D''=\frac{1}{V^{\frac{1}{2}}}\begin{vmatrix} k_{133}-H_3H_{31} & k_{12} & k_{13} \\ k_{233}-H_3H_{32} & H_2^2 & k_{23} \\ H_3H_{33} & k_{23} & H_3^2 \end{vmatrix}.$$

And $E=H_2^2$, $F=k_{23}$, $G=H_3^2$. These values substituted in (G_1) give the Gauss equation.

The equations (C) of course must be equivalent to the equations given by Codazzi, *Ann. di Mat.*, Ser. 3, Vol. II, 1869, p. 285. The connection can be obtained without very much reckoning by using the form of the Codazzi relations standing in Darboux, *Surfaces*, Vol. III, p. 248, (24) and (25). It is thus seen that this equation (24) taken for the surface $\rho_1=$const. is the combination of (G_1) and (C_2), and the equation (25) of (G_1) and (C_3).

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